

Is Wolfram and Cook's (2,5) Turing machine *really* universal?

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November 5, 2007

Abstract

Wolfram [2, p. 707] and Cook [1, p. 3] claim to prove that a (2,5) Turing machine (2 states, 5 symbols) is universal, via a universal cellular automaton known as *Rule 110*. The first part of this paper points out a critical gap in their argument. The second part bridges the gap, thereby giving what appears to be the first proof of universality.

1 The claim

Wolfram [2, p. 707] and Cook [1, p. 3] claim to prove that the Turing Machine M with the following table is universal:

		State	
		◦	•
Input symbol	0	◦ <u>0</u>	◦ <u>1</u>
	1	• <u>1</u>	• ?
	<u>0</u>	0 ◦	0 ◦
	<u>1</u>	1 •	1 •
	?	0 ◦	1 ◦

Here table entry “• ?” means “write ? and move left into state •”, entry “1 ◦” means “write 1 and move right into state ◦”, *etc.*¹ For ease of reference, we collect together the passages from these works which together constitute the claimed proof of universality:

1. **Wolfram, p. 707.** . . . by using the universality of rule 110 it turns out to be possible to come up with the vastly simpler Turing machine shown below—with just 2 states and 5 possible colors.
2. **Wolfram, p. 707, first caption.** The rule for the simplest Turing machine currently known to be universal, based on discoveries in this book. The machine has 2 states and 5 possible colors.
3. **Wolfram, p. 707, second caption.** An example of how the Turing machine above manages to emulate rule 110.
4. **Wolfram, p. 708.** As the picture at the bottom of the previous page illustrates, this Turing machine emulates rule 110 in a quite straightforward way: its head moves systematically backwards and forwards, at each complete sweep updating all cells according to a single step of rule 110 evolution. And knowing from earlier in this chapter that rule 110 is universal, it then follows that the 2-state 5-color Turing machine must also be universal.

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¹Cook [1] writes 0² for 0, 1² for ? and ≠ for 1. Wolfram on p. 707 [2] uses shades of grey (white, light grey, medium grey, dark grey, black for 0, 0, ?, 1, 1, resp.), enumerated 0 . . . 4 from light to dark on p. 1119. The states • ◦ are $S_E S_O$ (resp.) for Cook, and ♠ ♣ (resp.) for Wolfram. The reason for our notational choice becomes clear with the example in Section 3.

5. **Wolfram, p. 1119 (note to p. 707).** *Rule 110 Turing machines.* Given an initial condition for rule 110, the initial condition for the Turing machine shown here is obtained as *Prepend[list, 0]* with 0's on the left and 0's on the right.²
6. **Cook, p. 3.** ... we can construct Turing machines that are universal because they can emulate the behavior of Rule 110. These machines, shown in Figure 1, are far smaller than any previously known universal Turing machines.
7. **Cook, p. 4, caption to Figure 1.** *Some small Turing machines which are universal due to being able to emulate the behavior of Rule 110 by going back and forth over an ever wider stretch of tape, each time computing one more step of Rule 110's activity.*

This list is the full extent of the Wolfram-Cook universality argument (aside from Wolfram's depiction of an example run of the Turing machine, p. 707).³ They attempt to argue as follows:

- (1) The Turing machine M emulates the following cellular automaton R ("Rule 110"):

000	001	010	011	100	101	110	110
0	1	1	1	0	1	1	0

- (2) R is universal;

- (3) hence M is universal.

2 The gap

Unfortunately, there is a critical gap in their attempted argument. Wolfram (item 5 above) defines the emulation of R on initial configurations

$$\Leftarrow 0 \ I \ 0 \Rightarrow$$

where I is a word (finite sequence of 1's and 0's), and $\Leftarrow w$ (resp. $w \Rightarrow$) denotes the infinite repetition of a word or symbol w towards the left (resp. right). However, Wolfram and Cook demonstrated the universality of R via initial configurations

$$\Leftarrow X \ I \ Y \Rightarrow$$

for words X and Y , neither constantly 0. In the former, there is an infinite stream of 0's either side of the input I , hence a finite number of 1's in total. In the latter, there are infinitely many 1's either side of I . This breakdown in reasoning begs the question: is their (2, 5) Turing machine M *really* universal?⁴

²This item is not a strictly verbatim quote: to match our notation for the Turing machine symbols, we have substituted "list" for Wolfram's original "4 list", and "0's on the left" for "1's on the left". Wolfram uses the Turing machine tape symbol '4' (depicted as a solid black square) to correspond to the cellular automaton's 1, while we (like Cook) use the tape symbol '1'; where he writes the tape symbol '1' (depicted as a light grey square), we write '0' (Cook's '0²').

³If there are additional details somewhere in [1, 2], they are not easy to find. In addition, after extensive web search, we were unable to find a universality proof. Note: the reader should not confuse the *proof of universality of rule 110* in Cook [1], which is laid out in full, with the *claimed proof of universality of the (2,5) Turing machine*, the full extent of which is items 1–7 above.

⁴Naive attempts to bridge the gap from $\Leftarrow 0 \ I \ 0 \Rightarrow$ emulations to $\Leftarrow X \ I \ Y \Rightarrow$ emulations fail. For example, one could run the emulation on $\Leftarrow 0 \ X^n \ I \ Y^n \ 0 \Rightarrow$, where X^n denotes n repetitions of X , the idea being that X^n and Y^n might contain enough gliders/particles [2, 1] to complete the computation. However, because of the halting problem, we can never predict how large n will need to be. Accordingly, we could resort to repeating the $\Leftarrow 0 \ X^n \ I \ Y^n \ 0 \Rightarrow$ emulation again and again, with progressively larger n , since if the target computation on $\Leftarrow X \ I \ Y \Rightarrow$ completes, then some n will be large enough that $\Leftarrow 0 \ X^n \ I \ Y^n \ 0 \Rightarrow$ completes in a corresponding manner. Alas, the *deus ex machina* (repeatedly restarting the Turing machine) destroys any possible claim of universality.

3 The solution

This section bridges the critical gap in the Wolfram-Cook argument, yielding what appears to be the first proof of universality of their 2-color 5-symbol Turing machine M .

Recall (item 5 above) that Wolfram defined emulation on initial configurations

$$\Leftarrow 0 \ I \ 0 \Rightarrow$$

by running M on the initial tape

$$\Leftarrow \underline{0} \ 0 \ I \ 0 \Rightarrow$$

The initial state and position of the tape head is not specified. However, it is easy to see that it suffices to start in state \circ (Wolfram's \blacklozenge) on the cell immediately to the right of I :⁵

$$\Leftarrow \underline{0} \ 0 \ I \ \overset{\circ}{0} \ 0 \Rightarrow$$

M simulates the cellular automaton R thus: if $M(i)$ is the state of the tape when the head first reaches the i^{th} cell right of I (the cell immediately adjacent to I being counted as the 0^{th}), then the i^{th} state of the cellular automaton R is obtained from $M(i)$ by replacing $\underline{0}$'s by 0 's. For example, if I is 111011 then $M(0)$ to $M(3)$ are

$$\begin{aligned} &\Leftarrow \underline{0} \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ \overset{\circ}{0} \ 0 \Rightarrow \\ &\Leftarrow \underline{0} \ 0 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ \overset{\circ}{0} \ 0 \Rightarrow \\ &\Leftarrow \underline{0} \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ \overset{\circ}{0} \ 0 \Rightarrow \\ &\Leftarrow \underline{0} \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ \overset{\circ}{0} \ 0 \Rightarrow \end{aligned}$$

clearly emulating R . The head sweeps repeatedly left and right, reaching one cell further each time. It is instructive to see the tape at the end of each leftward sweep:

$$\begin{aligned} &\Leftarrow \underline{0} \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ \overset{\circ}{0} \ 0 \Rightarrow \\ &\Leftarrow \underline{0} \ \overset{\circ}{0} \ \underline{1} \ ? \ ? \ \underline{1} \ \underline{1} \ ? \ \underline{1} \ \underline{0} \ 0 \Rightarrow \\ &\Leftarrow \underline{0} \ 0 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ \overset{\circ}{0} \ 0 \Rightarrow \\ &\Leftarrow \underline{0} \ \overset{\circ}{0} \ \underline{1} \ ? \ \underline{1} \ \underline{1} \ ? \ ? \ ? \ \underline{1} \ \underline{0} \ \underline{0} \ 0 \Rightarrow \\ &\Leftarrow \underline{0} \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ \overset{\circ}{0} \ 0 \Rightarrow \\ &\Leftarrow \underline{0} \ \overset{\circ}{0} \ \underline{1} \ ? \ ? \ ? \ ? \ \underline{1} \ \underline{0} \ \underline{1} \ \underline{1} \ \underline{0} \ \underline{0} \ \underline{0} \ 0 \Rightarrow \\ &\Leftarrow \underline{0} \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ \overset{\circ}{0} \ 0 \Rightarrow \end{aligned}$$

Each leftward sweep tries to update a cell according to its right neighbour only; if the status of the cell cannot be determined, the head writes '?'. Each rightward sweep resolves the '?'s.⁶ The state acts as a single bit 'carry', memorising the relevant neighbour, and is set to \bullet after reading a 1 or $\underline{1}$.

Wolfram and Cook demonstrate the universality of the cellular automaton R by emulating a universal cyclic tag system U . Given an input word J to U , they compute a word $I = I(J)$ (a simple substitution of words for symbols) and run R with the initial configuration

$$\Leftarrow X \ I \ Y \Rightarrow$$

for particular words X and Y , seeding the infinite repetitions $\Leftarrow X$ and $Y \Rightarrow$. These words remain fixed for different I (possible since R emulates a fixed universal cyclic tag system).

⁵This is consistent with Wolfram's example depicted on p. 707.

⁶This interpretation clarifies why we chose the notation $\underline{0} \ ? \ \underline{1} \ 1$ for the symbols of the Turing machine.

$$\begin{array}{l} \Leftarrow X \sqsubseteq X \sqsubseteq I \sqsubseteq Y \sqsubseteq Y \Rightarrow \\ \text{emulates } R \text{ with initial state} \quad \Leftarrow X \sqsubseteq I \sqsubseteq Y \Rightarrow \end{array}$$
$$X = 111011 \qquad I = 10011 \qquad Y = 1101$$

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1 1 1 0 1 1 1 1 1 0 1 1 1 1 1 0 1 1 1 1 1 0 1 1 1 1 0 0 1 1 1 1 0 1 1 1 1 0 1 1 1 0 1 1 1 0 1 1 1 0 1
0 1 1 1 0 0 0 0 1 1 1 0 0 0 0 1 1 1 0 0 0 0 1 1 1 0 1 0 1 1 0 0 1 1 1 0 1 1 1 0 1 1 1 0 1 1 1 0 1 1
1 0 1 0 0 1 1 0 1 0 0 1 1 0 1 0 0 1 1 0 1 1 1 1 1 0 1 1 0 1 1 1 0 1 1 1 0 1 1 1 0 1 1 1 0 1 1 1 1
1 1 0 1 1 1 1 1 0 1 1 1 1 1 0 1 1 1 1 1 0 0 0 0 1 1 1 1 1 1 0 1 1 1 0 1 1 1 0 1 1 1 0 1 1 1 1 0
1 1 1 0 0 0 1 1 1 0 0 0 1 1 1 0 0 0 1 0 0 0 1 1 0 0 0 0 1 1 0 0 0 0 1 1 1 0 1 1 1 0 1 1 1 0 1 1 1 0 1
0 1 0 0 1 1 0 1 0 0 1 1 0 1 0 0 1 1 0 0 1 1 0 0 1 1 1 0 0 0 1 1 0 1 1 1 0 1 1 1 0 1 1 1 0 1 1 1 1
1 0 1 1 1 1 1 0 1 1 1 1 1 1 0 1 1 1 1 0 1 1 1 0 1 1 0 1 0 0 1 1 1 1 1 1 0 1 1 1 0 1 1 1 0 1 1 1 1
1 1 0 0 0 1 1 1 0 0 0 1 1 1 0 0 1 1 1 0 1 1 1 1 1 1 0 1 1 0 0 0 1 1 1 1 0 1 1 1 0 1 1 1 0 1 1 1 0
1 0 0 1 1 0 1 0 0 1 1 0 1 0 1 1 1 0 0 0 0 1 1 1 1 0 0 1 1 1 0 1 1 1 0 1 1 1 0 1 1 1 0 1 1 1 0 1
0 1 1 1 1 1 0 1 1 1 1 1 0 1 0 0 0 1 1 0 0 1 0 1 1 1 1 1 1 0 1 1 1 0 1 1 1 0 1 1 1 0 1 1
1 0 0 0 1 1 1 0 0 0 1 1 1 0 0 1 1 1 0 1 1 1 1 0 0 0 1 1 1 1 0 0 0 1 1 1 1 0 1 1 1
0 0 1 1 0 1 0 0 1 1 0 1 0 1 1 0 1 1 1 0 0 1 0 0 1 1 0 1 1 1 0
1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1 0 1 0 1 1 0 1 1 1 1 1 0 1

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$$X = 11\underline{1000}\underline{1011} \quad X \text{ empty} \quad Y = \underline{11} \quad Y^{\top} = \underline{01}$$
[illegible][illegible]

4

The head makes progressively larger left and right sweeps, analogous to Wolfram and Cook's limited emulation on $\Leftarrow 0 \ 0 \ 0 \Rightarrow$. The tape is shown at the end of every rightward sweep. Larger symbols emphasise the causal future of I . Note that, on the causal future of I , the Turing machine has indeed emulated the cellular automaton. Below we have interleaved the ends of the leftward sweeps:

[illegible]

In Wolfram and Cook’s proof of the universality of R , the state of the emulated universal cyclic tag system U is recovered from the causal future of the input I alone (possible since the update rule observes only nearest neighbours). Thus our emulation of R on the future of I suffices for universality of the Turing machine.

In overview, the wrap construction on X and Y will proceed as follows. Consider the case $X = 111011$ as in the example above. Run the *wrapped* form of the cellular automaton on X , that is, with just six cells (the length of X), and where the last cell is formally considered to be the left neighbour of the first. Truncate the computation just as a row is about to recur (in this case, the first row). The result is below-left:

- If $s = 0$, move the cursor to the cell to the right. Go to $\star\star$.
- $\star\star$ Let s be the symbol at the cursor. Write down \underline{s} (the underlined variant of s).
- If $s = 0$, write down $\underline{0}$ and move the cursor to the cell to the right. Go to $\star\star$.
- If $s = 1$, write down 1 and move the cursor to the cell which is one column to the right and one row down. Go to \star .

As before, we terminate when the cursor lands on a cell which has already been visited (in this case, the third cell, 0 , in the top row). Every time we write down a new symbol, we add it to the *right* of W . The resulting 6-symbol word W is:

$$\underline{1}\underline{1}\underline{0}1\underline{0}1$$

The first two symbols $\underline{1}\underline{1}$ in W came from the two 1's at the beginning of the top row in the 2×4 matrix. These two 1's are not part of the cycle of the cursor: were we to continue making cursor moves, we would never revisit them. This part of W becomes Y_{\downarrow} , called the **right stem**:

$$Y_{\downarrow} = \underline{1}\underline{1}$$

The shortest word whose repetition yields the remainder $\underline{0}1\underline{0}1$ of W becomes Y^{\neg} , the **right seed**:⁹

$$Y^{\neg} = \underline{0}1$$

Observe that the initial tape to the right of the input I in the emulation above (p. 4) is the right stem Y_{\downarrow} followed by the infinite repetition of the right seed Y^{\neg} .

The construction described here does not work for words X and Y with the property that one of the rows in our constructed matrix consists of 0s only. For those words a simpler construction is possible. We do not describe this simpler construction here, as the words X and Y used by Wolfram and Cook in the simulation of U by R have the property that such a row of 0s will not emerge.

4 Formal details

Let $\mathbb{N} = \{0, 1, \dots\}$ and $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$.

Words. Let Σ be a set of *symbols*. A **word over** Σ , or Σ -**word**, of length $k \in \mathbb{N}$ is a function $w : \{0, \dots, k-1\} \rightarrow \Sigma$. We shall often write the sequence $w(0)w(1) \cdots w(k-1)$ of outputs of w , in order, to denote w . For example, 1011 denotes the $\{0, 1\}$ -word w of length 4 with $w(0) = w(2) = w(3) = 1$ and $w(1) = 0$. A length 0 word is **empty**. Define the **reverse** w^{rev} of w , also of length k , by $w^{\text{rev}}(i) = w(k-1-i)$. For example $(11010)^{\text{rev}} = 01011$. A Σ -word w of length k **contains** $x \in \Sigma$ if $w(i) = x$ for some $i \in \{0, \dots, k-1\}$.

Let v and w be Σ -words of length k and l , respectively. The **concatenation** $v;w$ of v and w is the Σ -word of length $k+l$ defined by $(v;w)(i) = v(i)$ for $0 \leq i < k$ and $(v;w)(k+j) = w(j)$ for $0 \leq j < l$. For example, if $v = 1011$ and $w = 00001$ then $v;w = 101100001$. Since $u;(v;w) = (u;v);w$ we can write $u;v;w$ without ambiguity. For a Σ -word w of length k and $0 \leq a < k$ define $w_{<a}$ as the restriction of w to $\{0, \dots, a-1\}$ (a Σ -word of length a) and define $w_{\geq a}$ as the remainder: the unique Σ -word v such that $w = w_{<a};v$ (a Σ -word of length $k-a$).

⁹The **left stem** $\lrcorner X$ was empty (since the cursor returned to the top-right cell of the 9×6 grid, on which it started), therefore we simplified the exposition by not mentioning it. In general, however, there may be a non-empty left stem.

Write Σ^* for the set of Σ -words. Given a Σ^* -word W of length k (a “word of words”) its **flattening** W^\flat is the Σ -word $W(0);W(1);\dots;W(k-1)$. For example, if $W = (111)(00)(1101)$ (the $\{0,1\}^*$ -word of length 3 given by $W(0) = 111$, $W(1) = 00$ and $W(2) = 1101$) then $W^\flat = 111001101$. The **n -fold repetition** w^n of w is empty if $n = 0$ and $w;w^{n-1}$ if $n > 0$ (i.e., $w;w;\dots;w$ with n occurrences of w). The **reduction** $|w|$ of w , when it exists, is the shortest (minimal length) word r such that $w = r^n$ for some $n \geq 1$. For example, $|110110110110| = 110$.

State. A **state over** Σ , or **Σ -state**, is a function $S : \mathbb{Z} \rightarrow \Sigma$. Each c in the domain \mathbb{Z} of S is a **cell**. Let A, I, B be a Σ -words of lengths a, l, b respectively. Define

$$\Leftarrow A \ I \ B \Rightarrow$$

as the Σ -state $S : \mathbb{Z} \rightarrow \Sigma$ comprising I on cells 0 to $l-1$, infinite repetitions of A to the left, and infinite repetitions of B to the right:

$$S(c) = \begin{cases} I(c) & \text{if } 0 \leq c < l \\ A(c \% a) & \text{if } c < 0 \\ B((c-l) \% b) & \text{if } c \geq l \end{cases}$$

Given additional Σ -words P and Q of lengths p and q , respectively, define

$$\Leftarrow A \ P \ I \ Q \ B \Rightarrow$$

as the Σ -state $T : \mathbb{Z} \rightarrow \Sigma$ obtained by inserting P and Q either side of I in $\Leftarrow A \ I \ B \Rightarrow$:

$$T(c) = \begin{cases} I(c) & \text{if } 0 \leq c < l \\ A((c+p) \% a) & \text{if } c < -p \\ B((c-l-q) \% b) & \text{if } c \geq l+q \\ P(c+p) & \text{if } -p \leq c < 0 \\ Q(c-l) & \text{if } l \leq c < l+q \end{cases}$$

4.1 The rule 110 cellular automaton R

Define **spacetime** as the product $\mathbb{N} \times \mathbb{Z}$ where \mathbb{N} is the set of **times** and \mathbb{Z} is the set of **cells** (space). Each ordered pair $(t, c) \in \mathbb{N} \times \mathbb{Z}$ is an **event** (spacetime coordinate). A **run** of R is a function $\rho : \mathbb{N} \times \mathbb{Z} \rightarrow \{0, 1\}$ such that for all times $t \in \mathbb{N}$ and cells $c \in \mathbb{Z}$ the following condition holds:

- **Causality.** $\rho(t+1, c) \neq \rho(t, c)$ if and only if:
 - (Birth¹⁰) $\rho(t, c) = 0$ and $\rho(t, c+1) = 1$, or
 - (Death¹¹) $\rho(t, c-1) = \rho(t, c) = \rho(t, c+1) = 1$.

For $t \in \mathbb{N}$ the t^{th} **state** or **state at time** t of a run ρ , denoted ρ_t , is the $\{0, 1\}$ -state $\rho_t : \mathbb{Z} \rightarrow \{0, 1\}$ given by $\rho_t(c) = \rho(t, c)$ for all cells $c \in \mathbb{Z}$. The **initial state** of ρ is ρ_0 . Note that R is deterministic: the state ρ_t at each time $t \in \mathbb{N}$ is determined by the initial state ρ_0 .

¹⁰“A 1 is born from a 0 whose right neighbour is 1,” visually $\begin{smallmatrix} 01 \\ 1 \end{smallmatrix}$.

¹¹“A 1 dies by overcrowding when both neighbours are 1,” visually $\begin{smallmatrix} 111 \\ 0 \end{smallmatrix}$.

Wrapped rule 110. For $n \in \mathbb{N}$ define n -**wrapped spacetime** as $\mathbb{N} \times \{0, \dots, n-1\}$. An n -**wrapped run** of the cellular automaton R is a function $\rho : \mathbb{N} \times \{0, \dots, n-1\} \rightarrow \{0, 1\}$ which, for all times $t \in \mathbb{N}$ and cells $c \in \{0, \dots, n-1\}$, satisfies the *Causality* condition defined above upon interpreting $\rho(t, n)$ as $\rho(t, 0)$ and $\rho(t, -1)$ as $\rho(t, n-1)$. A 6-wrapped run ρ is depicted below-left.

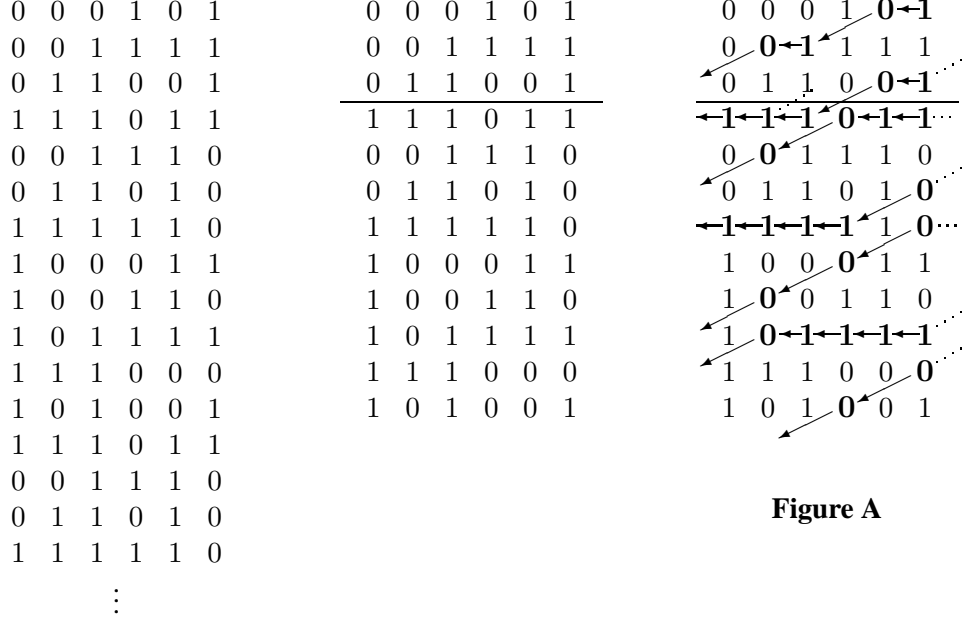


Figure A

The t^{th} **word** of an n -wrapped run ρ , denoted ρ_t , is given by $\rho_t(c) = \rho(t, c)$ for all $c \in \{0, \dots, n-1\}$. The **initial word** of ρ is ρ_0 . Wrapped runs are deterministic: ρ_t is determined at each time t by the initial word ρ_0 . The figure above-left shows words ρ_0 to ρ_{15} , from top to bottom.

The **onset of periodicity** in ρ is the least time $\alpha \in \mathbb{N}$ such that $\rho_\alpha = \rho_{\alpha+\delta}$ for some $\delta > 0$.¹² The least such δ is the **period** of ρ , and $\alpha + \delta$ is the **time of first repetition**. In the example above-left, $\alpha = 3$ and $\delta = 9$ ($\rho_3 = \rho_{3+9} = 111011$).

The following lemma is trivial, but we state and prove it properly nonetheless.

LEMMA 1 (PERIODICITY) *Let ρ be an n -wrapped run with period δ and onset of periodicity α . Then $\rho_{t+\delta} = \rho_t$ for all $t \geq \alpha$.*

Proof. By induction on t . Induction base: the condition holds for $t = \alpha$, by definition of α . Induction step: if $\rho_t = \rho_u$ then $\rho_{t+1} = \rho_{u+1}$, by the *Causality* condition defining an n -wrapped run. \square

4.2 The wrap constructions

The left wrap construction. Let ρ be an n -wrapped run with period δ and onset of periodicity α . Let $\beta = \alpha + \delta$, the time of first repetition. The **matrix** $\hat{\rho}$ of ρ is the restriction of ρ to times prior to β , i.e., the restriction of ρ to the domain $\{0, \dots, \beta-1\} \times \{0, \dots, n-1\}$. For the example ρ depicted above-left, the matrix $\hat{\rho}$ is shown above-centre. The horizontal rule is a visual aid to emphasise the onset of periodicity $\alpha = 3$ and the period $\delta = 9$.

¹²The onset of periodicity α exists since there are at most 2^n distinct $\{0, 1\}$ -words of length n .

Let ρ be an n -wrapped run with period δ and onset of periodicity α . Let $\beta = \alpha + \delta$, the time of first repetition. Write E for the domain $\{0, \dots, \beta - 1\} \times \{0, \dots, n - 1\}$ of the matrix $\hat{\rho}$. A **trajectory** in $\hat{\rho}$ is a function (infinite sequence) $\mathbb{N} \rightarrow E$. The **left wrap trajectory** of ρ is the trajectory \overleftarrow{T}_ρ in $\hat{\rho}$ defined by the following recursion:¹³

- $\overleftarrow{T}_\rho(0) = (0, n - 1)$ ¹⁴
- if $\overleftarrow{T}_\rho(i) = (t, c)$ then $\overleftarrow{T}_\rho(i + 1) = \begin{cases} (t, (c - 1) \% n) & \text{if } \hat{\rho}(t, c) = 1 \\ (\alpha + [(t - \alpha + 1) \% \delta], (c - 2) \% n) & \text{if } \hat{\rho}(t, c) = 0 \end{cases}$

where $x \% y$ is the remainder upon dividing x by y , i.e., the unique $z \in \{0, \dots, y - 1\}$ such that $z = x + yk$ for some $k \in \mathbb{Z}$. The left wrap trajectory \overleftarrow{T}_ρ of our running example ρ is shown top-right (Figure A): $\overleftarrow{T}_\rho(0)$ to $\overleftarrow{T}_\rho(6)$ are, in order, $(0, 5)$, $(0, 4)$, $(1, 2)$, $(1, 1)$, $(2, 5)$, $(2, 4)$, $(3, 2)$. Note the ‘wrap’ from $\overleftarrow{T}_\rho(27) = (11, 3)$ (the bottom row) to $\overleftarrow{T}_\rho(28) = (3, 2)$ (the row below the horizontal rule marking the onset of periodicity $\alpha = 3$).

The **start of cyclicity** in the left wrap trajectory \overleftarrow{T}_ρ is the least index a such that $\overleftarrow{T}_\rho(a) = \overleftarrow{T}_\rho(a + d)$ for some $d > 0$. The least such d is the **period** of the trajectory, and $a + d$ is the **index of first recurrence**. In the running example (Figure A), $a = 7$ and $d = 21$ ($\overleftarrow{T}_\rho(7) = \overleftarrow{T}_\rho(7 + 21) = (3, 1)$).

Write $\overleftarrow{\rho}$ for the composite of ρ and \overleftarrow{T}_ρ defined by $\overleftarrow{\rho}(i) = \rho(\overleftarrow{T}_\rho(i))$, listing the symbols along the trajectory \overleftarrow{T}_ρ . In our example (Figure A), $\overleftarrow{\rho}(0)\overleftarrow{\rho}(1)\overleftarrow{\rho}(2) \dots$ is (from left to right)

$$10101011111000111100011110001111000 \dots$$

obtained by simply reading the symbols encountered along the arrows.

For $c \in \{0, \dots, n - 1\}$ define the n -**wrapped decrement** c^- as $c - 1$ if $c > 0$ and n if $c = 0$.¹⁵ For $(t, c) \in \mathbb{N} \times \{0, \dots, n - 1\}$ define $(t, c)^- = (t, c^-)$. Define $\overleftarrow{\rho}^-$ by $\overleftarrow{\rho}^-(i) = \rho(\overleftarrow{T}_\rho(i)^-)$, listing the (wrapped-)left neighbours of the symbols in the trajectory \overleftarrow{T}_ρ . In our example (Figure A), $\overleftarrow{\rho}^-(0)\overleftarrow{\rho}^-(1)\overleftarrow{\rho}^-(2) \dots$ is $010000111101011 \dots$.

Let X be a $\{0, 1\}$ -word. We shall define $\{0, \underline{0}, \underline{1}, 1\}$ -words $\ulcorner X$ and $\lrcorner X$ called the **left seed** and the **left stem**, respectively. Let ρ be the n -wrapped run with $\rho_0 = X$. Let d be the period of the left wrap trajectory \overleftarrow{T}_ρ , let a be its start of cyclicity, and let $b = a + d$, the index of first recurrence. Let T be the restriction of \overleftarrow{T}_ρ to the domain $\{0, \dots, b - 1\}$. Define the $\{1, 0\underline{0}, 0\underline{1}\}$ -word U of length b by

$$U(i) = \begin{cases} 1 & \text{if } \overleftarrow{\rho}(i) = 1 \\ 0 \star & \text{if } \overleftarrow{\rho}(i) = 0, \text{ where } \star = \overleftarrow{\rho}^-(i) \end{cases}$$

In our running example (Figure A), U is

$$1(0\underline{1})1(0\underline{0})1(0\underline{0})1 \quad 1111(0\underline{1})(0\underline{0})(0\underline{1}) \quad 1111(0\underline{1})(0\underline{0})(0\underline{1}) \quad 1111(0\underline{1})(0\underline{0})(0\underline{1})$$

of length 28, where gaps highlight repetition. Recall that a is the start of cyclicity. Define the left stem $\lrcorner X = (U_{<a})^{\text{rev}}$ (the reverse of the flattening of $U_{<a}$) and the left seed $\ulcorner X = |(U_{\geq a})^{\text{rev}}|$ (the reduction of the reverse of the flattening of $U_{\geq a}$). In our example, $a = 7$, so $U_{<a} = 1(0\underline{1})1(0\underline{0})1(0\underline{0})1$ and $U_{\geq a} = [1111(0\underline{1})(0\underline{0})(0\underline{1})]^3$, hence

$$\lrcorner X = \underline{1001001101} \quad \ulcorner X = \underline{1000101111}$$

¹³See page 6 for a more informal presentation.

¹⁴The top-right cell of the matrix.

¹⁵I.e., $c^- = (c - 1) \% n$.

The right wrap construction. The *right wrap trajectory* in the matrix $\hat{\rho}$ is the function (infinite sequence) $\vec{T}_\rho : \mathbb{N} \rightarrow E$ defined by the following recursion.¹⁶ Recall that δ is the period and α is the time of first repetition.

- $\vec{T}_\rho(0) = (0, 0)$;
- if $\vec{T}_\rho(i) = (t, c)$ then $\vec{T}_\rho(i+1) = \begin{cases} (t, (c+1)\%n) & \text{if } \hat{\rho}(t, c) = 1 \\ (\alpha + [(t - \alpha + 1)\%\delta], (c+2)\%n) & \text{if } \hat{\rho}(t, c) = 0 \end{cases}$

The *start of cyclicity*, *period* and *index of first recurrence* are defined as for the left wrap trajectory. Write $\vec{\rho}$ for the composite of ρ and \vec{T}_ρ defined by $\vec{\rho}(i) = \rho(\vec{T}_\rho(i))$, listing the symbols along the trajectory \vec{T}_ρ . For $c \in \{0, \dots, n-1\}$ define the *wrapped increment* c^+ as $c+1$ if $c < n$ and 0 if $c = n$. For $(t, c) \in \mathbb{N} \times \{0, \dots, n-1\}$ define $(t, c)^+ = (t, c^+)$. Define $\vec{\rho}^+$ by $\vec{\rho}^+(i) = \rho(\vec{T}_\rho(i)^+)$, listing the (wrapped-)right neighbours of the symbols in the trajectory \vec{T}_ρ .

Let Y be a $\{0, 1\}$ -word. We shall define $\{0, \underline{0}, \underline{1}, 1\}$ -words Y^\frown and Y_\frown called the *right seed* and the *right stem*, respectively. Let ρ be the n -wrapped run with $\rho_0 = Y$. Let d be the period of the right wrap trajectory \vec{T}_ρ , let a be its start of cyclicity, and let $b = a + d$, the index of first recurrence. Let T be the restriction of \vec{T}_ρ to the domain $\{0, \dots, b-1\}$. Define the $\{\underline{1}, \underline{00}, \underline{01}\}$ -word U of length b by

$$U(i) = \begin{cases} \underline{1} & \text{if } \vec{\rho}(i) = 1 \\ \underline{0}\star & \text{if } \vec{\rho}(i) = 0, \text{ where } \star = \vec{\rho}^+(i) \end{cases}$$

Define the right stem by $X_\frown = (U_{<a})^\frown$ (the flattening of $U_{<a}$) and the right seed by $X^\frown = |(U_{\geq a})^\frown|$ (the reduction of the flattening of $U_{\geq a}$).

4.3 The Wolfram-Cook Turing machine M

Let $\Sigma_M = \{0, \underline{0}, ?, \underline{1}, 1\}$. A *configuration* $\langle \tau, h, q \rangle$ of the Wolfram-Cook Turing machine M is a Σ_M -state τ (i.e., a function $\tau : \mathbb{Z} \rightarrow \{0, \underline{0}, ?, \underline{1}, 1\}$) called the *tape*, a *head position* $h \in \mathbb{Z}$, and a *head state* $q \in \{\circ, \bullet\}$. Write \mathcal{C} for the set of all configurations.¹⁷ Write $\langle \tau, h, q \rangle \rightsquigarrow \langle \tau', h', q' \rangle$ if the following conditions are satisfied (formalising the transition table on page 1):

- $\tau'(c) = \tau(c)$ for all $c \neq h$;¹⁸
- if $\tau(h) \in \{0, 1\}$ then $h' = h - 1$, otherwise $h' = h + 1$;¹⁹
- if $\tau(h) \in \{1, \underline{1}\}$ then $q' = \bullet$, otherwise $q' = \circ$;²⁰
- if $q = \circ$ then the ordered pair $\langle \tau(h), \tau'(h) \rangle$ is in $\{\langle 0, \underline{0} \rangle, \langle 1, \underline{1} \rangle, \langle \underline{0}, 0 \rangle, \langle \underline{1}, 1 \rangle, \langle ?, 0 \rangle\}$;²¹
- if $q = \bullet$ then the ordered pair $\langle \tau(h), \tau'(h) \rangle$ is in $\{\langle 0, \underline{1} \rangle, \langle 1, ? \rangle, \langle \underline{0}, 0 \rangle, \langle \underline{1}, 1 \rangle, \langle ?, 1 \rangle\}$.²²

¹⁶See page 6 for a more informal presentation.

¹⁷Thus $\mathcal{C} = \{0, \underline{0}, ?, \underline{1}, 1\}^{\mathbb{Z}} \times \mathbb{Z} \times \{\circ, \bullet\}$ where X^Y denotes the set of all functions from Y to X .

¹⁸The head writes to cell h only; all other cells remain the same.

¹⁹The head moves left from 0 and 1, and right otherwise.

²⁰The next state is \bullet ('carry') iff the head reads 1 or $\underline{1}$.

²¹Corresponding to the first column in the table on page 1.

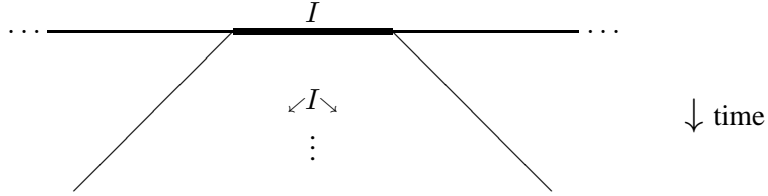
²²Corresponding to the second column in the table on page 1.

M is deterministic: for any configuration C there is a unique configuration C' such that $C \rightsquigarrow C'$. A **run** of M is a function $\mu : \mathbb{N} \rightarrow \mathcal{C}$ such that $\mu(t) \rightsquigarrow \mu(t+1)$ for all t . By determinism of M , the run is uniquely determined by the **initial configuration** $\mu(0)$. The **initial tape**, **initial head position** and **initial head state** are the tape, head position and head state of the initial configuration. Let μ be a run of M , and define the functions τ_μ , h_μ and q_μ by $\mu(t) = \langle \tau_\mu(t), h_\mu(t), q_\mu(t) \rangle$, the first of which is the **tape function**. For $t > 0$, if $h_\mu(t+1) = h_\mu(t-1) = h_\mu(t) - 1$ the head **switches left at time** t and if $h_\mu(t+1) = h_\mu(t-1) = h_\mu(t) + 1$ the head **switches right at time** t . The head **switches left at time** 0 if $h_\mu(1) = h_\mu(0) - 1$ (the first head move is to the left). The configurations $\mu(t)$ of a run μ are depicted on page 4, for t such that the head switches left at time t . The next figure (page 5) in addition shows the configurations $\mu(t)$ for t such that the head switches right at time t .

Let the set L comprise those $t \in \mathbb{N}$ such that the head of μ switches left at time t , and write $L(i)$ for the i^{th} largest element of L (its smallest element taken to be $L(0)$, the 0^{th}). The **emulation** $\hat{\mu}$ produced by a run μ of M is the result of restricting the tape to times when the head switches left, and reindexing the timestamps to be sequential from 0. Formally (recalling that $\tau_\mu : \mathbb{Z} \rightarrow \{0, \underline{0}, ?, \underline{1}, 1\}$ denotes the tape function of μ), $\hat{\mu}(t) = \tau_\mu(L(t)) : \mathbb{Z} \rightarrow \{0, \underline{0}, ?, \underline{1}, 1\}$.²³ The lower figure on page 4 shows the sequence $\hat{\mu}(0), \hat{\mu}(1), \dots, \hat{\mu}(12)$.

4.4 The Causal Future Emulation Theorem

For any $\{0, 1\}$ -word I of length l , a **run from** I of the rule 110 cellular automaton R is a run ρ whose initial configuration has I in cells $0, \dots, l-1$, i.e., $\rho_0(c) = I(c)$ for $c \in \{0, \dots, l-1\}$. The **causal future** $\swarrow I \searrow$ of a run from I is the set of pairs $\{ \langle t, c \rangle : -t \leq c < k+t \} \subseteq \mathbb{N} \times \mathbb{Z}$.²⁴



In Wolfram and Cook's proof of the universality of R , the state of the emulated universal cyclic tag system on (transformed) input I is recovered from the future $\swarrow I \searrow$ alone. We shall refer to this property as **future sufficiency**.

Analogously, for the Turing machine M , a **run of from** I is a run μ of M whose initial tape has I in cells 0 to $l-1$, in other words, $\tau_\mu(0, c) = I(c)$ for $c \in \{0, \dots, l-1\}$.

The following theorem allows us to emulate the causal future $\swarrow I \searrow$ in the rule 110 automaton R on the Turing machine M .

THEOREM 1 (CAUSAL FUTURE EMULATION) *Let A, I, B be $\{0, 1\}$ -words of lengths a, l, b , respectively, with A and B each containing 0. Let ρ be the run of the rule 110 cellular automaton R from I with initial state*

$$\Leftarrow A I B \Rightarrow$$

Let μ be the run of the Wolfram-Cook Turing machine M from I with initial tape²⁵

$$\Leftarrow \lceil A \quad \lceil A \quad I \quad B \rceil \quad B \rceil \Rightarrow$$

²³Note that $\hat{\mu}(i)$ may be undefined for some i , for example if the head “runs off to infinity” at some point.

²⁴Note that this depends only on the integer l .

²⁵Recall that $\lceil A, \lceil A, B \rceil$ and $B \rceil$ were defined by the wrap constructions.

and head in initial state \circ and initial position l (the cell immediately to the right of I). Recall that $\hat{\mu}(i)$ denotes the tape function $\tau_{\mu}(t) : \mathbb{Z} \rightarrow \{0, \underline{0}, ?, \underline{1}, 1\}$ for the time t of the i^{th} occasion the head switches left. Then for all events $\langle t, c \rangle$ in the causal future $\prec I_{\searrow}$,

$$\rho(t, c) = (\hat{\tau}(t))(c) .$$

The proof of the theorem is the subject of the next section. Universality is an immediate corollary:

THEOREM 2 (UNIVERSALITY) *Wolfram and Cook’s 2-state 5-symbol Turing machine is universal.*

Proof. Take A and B in the Causal Future Emulation Theorem to be the infinitely repeated words used by Wolfram and Cook to simulate a cyclic tag system in the rule 110 automaton R [1, 2]. Universality follows from future sufficiency. \square

5 Proof of the Causal Future Emulation Theorem

Before working through this proof, the reader may wish to study the figure on page 5 which shows a causal future emulation at both left- and right switches of the head.

[...]

Acknowledgement

Many thanks to Rob van Glabbeek for detailed feedback. Thanks also to Vaughan Pratt.

References

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- [2] Stephen Wolfram (2002), *A New Kind of Science*, Wolfram Media, ISBN 1579550088.